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Discrete Nonlinear Approximation

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INTRODUCTION

In the application of nonlinear approximation theory it is often the case that a discrete problem is more difficult to treat than a continuous one. This is primarily due to the fact that the pertinent discontinuous functions are elements of the pointwise closure. In this study we will analyze such functions for several important families. These families, for the continuous case, were first introduced by Hobby and Rice [1], and were later studied by de Boor [2], and Barrar and Loeb [3, 4].

A typical family to be considered is generated by a function $\gamma(t, x)$ from $T \times [0, 1]$ to the reals where T is a subset of the real line. For a fixed positive integer N, we set

$$F = \left\{ f(x) = \sum_{i=1}^{N} a_i \gamma(t_i, x) \colon a_i \text{ real};\ t_i \in T \right\}.$$

For a sufficiently dense finite subset of [0, 1] we will be interested in examining the pointwise closure of F, which is an existence set in the sense that each real-valued function defined on this subset has a closest point in the closure. Indeed, using difference equation techniques we will be able to explicitly determine these pointwise closures for such important families as the exponentials and the rational functions generated by the Cauchy Kernel [6]. In order to simplify notation we restrict ourselves to equally spaced discrete subsets; i.e., a typical subset is generated by a number h which is the reciprocal of a positive integer. The subset which is labeled $[h]$ is of the form

$$[h] = \{0, h, 2h, \ldots, 1 - h, 1\}.$$

We will sometimes call $[h]$ a grid. The results in this paper can be extended to the unequally spaced case. The later sections of this paper will be devoted

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to the existence question for the continuous case using the L_p norms ($1 \leqslant p \leqslant \infty$). The discrete analogs of these norms will be employed to settle this continuous existence question by letting $h \to 0$.

The following notation will be helpful. If G is any subset of $C[0, 1]$, then $G[h]$ is the set of all such functions restricted to $[h]$; i.e.,

$$G[h] = \{g\colon [h] \to \text{Reals: for some } f \in G,\ g(x) = f(x), \text{ for } x \in [h]\}.$$

To analyze the existence problem for families of this type we shall make several assumptions on $\gamma(t, x)$. Although these assumptions are somewhat involved to state precisely, the principles behind them are quite natural.

For example, consider a sequence $f_\nu(x) = \sum_{i=1}^{N} a_{i\nu}\gamma(t_{i\nu}, x)$ in F. It may happen that for some $\bar{t}$, $t_{i\nu} \to \bar{t}$, $i = 1,\ldots, N$, where $\bar{t}$ may not be in T if T is not compact. (note: $\bar{t} = \pm\infty$ is allowed).

Thus, it is necessary to describe the types of limit functions that can arise when such a situation occurs. The case $\bar{t} \in T$ has been resolved (see [2] and Theorem 1).

It is natural to distinguish the case $\bar{t} \in T$ and $\bar{t} \notin T$. Moreover, it turns out to be desirable to classify the points arising in the latter situation. This is done as follows.

A "regular" point will be one which after a suitable change of variable can be treated as an element of a new parameter set where the problem is solved as in the first case. Any point for which this cannot be done will be called "singular."

Thus, we are led to examine three types of behavior. Assumptions 1 and 2 treat the case $\bar{t} \in T$ (see Theorem 1), and Assumptions 3 and 4 along with Definition 1 treat the case $\bar{t} \notin T$. A most important feature of these assumptions is that the functions arising from these types of behavior are to be independent of each other in the manner prescribed below. As will be seen, this allows them to be considered separately.

ASSUMPTION 1. *$\gamma^{(j)}(t, x) \equiv (\partial^j/\partial t^j)\,\gamma(t, x)$ is continuous for*

$$(t, x) \in T \times [0, 1] \quad \textit{and} \quad 0 \leqslant j \leqslant N - 1.$$

ASSUMPTION 2. *For h sufficiently small any function of the form*

$$\sum_{i=1}^{k} \sum_{j=1}^{m_i} a_{ij}\gamma^{(j)}(t_i, x) \tag{1}$$

where $\sum\sum |a_{ij}| > 0$ $t_i \in T$ and $\sum_{i=1}^{k} (m_i + 1) \leqslant N$, is not identically zero over $[h]$.

ASSUMPTION 3. *If $\bar{T}$ represents the closure of T, $\bar{T} \sim T$ consists of a finite number of points which are either regular or singular points. The set of regular points will be labeled R.*

To further introduce the concept of a "regular" point before proceeding with the formal definition, consider the following example. Let $\gamma(t, x) = 1/(x + t)$ where $T = (0, \infty)$ ($\gamma(t, x)$ is called the Cauchy kernel). Suppose that $\{f_\nu(x) = \sum_{i=1}^{K} (a_{i\nu}/(x + t_{i\nu})),\ K \leqslant N\}$ is a sequence in F such that $\{t_{i\nu}\} \to \infty$ for all i.

We can reparameterize the family F by defining $\lambda(t) = 1/t$ and defining a new generating function by $\Phi(\lambda, x) \equiv (1/\lambda)\, \gamma(1/\lambda, x) = 1/(\lambda x + 1)$ where $\lambda \in (0, \infty)$. Then we can extend the parameter set by defining $\Phi(0, x) = \lim_{\lambda \to 0} \Phi(\lambda, x) = 1$. Under this transformation the sequence $f_\nu(x)$ can be written as $\sum_{i=1}^{K} (\lambda_{i\nu} a_{i\nu}/(\lambda_{i\nu} x + 1)) \equiv \sum_{i=1}^{K} b_{i\nu} \Phi(\lambda_{i\nu}, x)$ where $\lambda_{i\nu} \to 0$.

Thus the problem is reduced to a coalescing problem in which the limit point lies in the parameter set. Parts (c) and (d) below will be verified later in this paper. Also it will be shown that "0" is a singular point for the Cauchy kernel.

DEFINITION 1. A $\bar{t} \in \bar{T} \sim T$ is called a *regular point* if there is a one-to-one real-valued function $\lambda(t)$ defined on a neighborhood $W(\bar{t})$ of $\bar{t}$ in T and a non-zero real-valued function $h(t)$ defined on the same neighborhood such that if we define

$$\varphi(\lambda, x) \equiv h(t(\lambda))\, \gamma(t(\lambda), x)$$

for $t \in W(\bar{t})$ (where $t(\lambda)$ is the inverse function for $\lambda(t)$) then over $W(\bar{t})$:

(a) $\lim_{t \to \bar{t}} \lambda(t) = \bar{\lambda}$;

(b) $\varphi(\lambda, x)$ can be extended to $\bar{\lambda}$ by the formula $\varphi(\bar{\lambda}, x) = \lim_{\lambda \to \bar{\lambda}} \varphi(\lambda, x)$;

(c) at $\bar{\lambda}$, $(\partial^j/\partial\lambda^j)\, \varphi(\lambda, x) \equiv \varphi^{(j)}(\lambda, x)$ is continuous in λ and x ($j = 0, 1, \ldots, N - 1$) for $x \in [0, 1]$;

(d) for sufficiently small h, any function of the form,

$$\sum_{j=0}^{m} a_j \varphi^{(j)}(\bar{\lambda}, x) \tag{2}$$

where $\sum |a_j| > 0$, is not identically zero over $[h]$. Further over $[h]$ such a function cannot be expressed as

$$g(x) + f(x),$$

where $g \in G_j$ and f is of the form

$$\sum_{i=1}^{k} \sum_{j=0}^{m_i} a_{ij} \varphi^{(j)}(\lambda_i, x),$$

with $\lambda_i \in R \sim \{\bar{\lambda}\}$ and $\sum_{i=1}^{k} (m_i + 1) + (m + 1) \leqslant N$. (When no confusion arises we call λ_i and $\bar{\lambda}$ regular points.) Here

$$G_j = \left\{ \sum_{i=1}^{s} \sum_{j=0}^{m_i} a_{ij} \gamma^{(j)}(t_i, x) \colon t_i \in T, \sum_{i=1}^{s} (m_i + 1) \leqslant j \right\}.$$

We further define

$$H_k = \left\{ \sum_{i=1}^{r} \sum_{j=0}^{m_i} a_{ij} \varphi^{(j)}(\lambda_i, x) \colon \lambda_i \in R; \sum_{i=1}^{r} (m_i + 1) \leqslant k \right\} \qquad (k = 1, \dots, N),$$

For example for the Cauchy Kernel H_k consists of all polynomials of degree at most $k - 1$.

Let S be the complement of $T \cup R$ in $\bar{T}$; that is,

$$S = \bar{T} \sim (T \cup R).$$

A point of S will be called a singular point and we further assume the folllowing.

Assumption 4. *For small h, if a sequence $\{\sum_{i=1}^{l} a_{i\nu} \gamma(t_{i\nu}, x)\} \subset F[h]$ converges pointwise to a nonzero function $f_S(x)$ over $[h]$ where $\lim_{\nu \to \infty} t_{i\nu} = t_i \in S$ $\{i = 1, \dots, l \leqslant N\}$ then $f_S(x)$ cannot be expressed as $g + h$ where $g \in G_j[h]$, $h \in H_k[h]$ and $j + k \leqslant N$.*

We will call such a $f_S(x)$ a singular function on the grid $[h]$.

Discrete Approximation

We will need the following Theorem which is a slight variation of a result proved in [2].

Theorem 1. *Consider any norm on G_N which is dominated by the uniform norm. Then if the sequence $\{f_\nu(x) \equiv \sum_{i=1}^{k} a_{i\nu} \gamma(t_{i\nu}, x)\} \subset F$ is bounded in this norm where $\lim_{\nu \to \infty} t_{i\nu} = t \in T$ $(i = 1, \dots, k)$ then there is a subsequence which converges in the uniform norm to an element of the form*

$$\sum_{i=0}^{k-1} a_i \gamma^{(i)}(t, x).$$

(*Here by a norm* $\| \ \|$ *on* G_N *we mean it is a semi-norm over* $C[0, 1]$ *and has the property that for any nonzero* $g \in G_N$ $\| g \| > 0$.)

COROLLARY 1. *For h small enough, any norm on the space of all real-valued functions on* $[h]$ *has the property: If the sequence*

$$\left\{ f_\nu(x) \equiv \sum_{i=1}^{k} a_{i\nu} \gamma(t_{i\nu}, x) \right\} \subset F$$

is bounded in this norm where $\lim_{\nu \to \infty} t_{i\nu} = t \in T$ $(i = 1,..., k)$ *then there is a subsequence with converges in the continuous uniform norm to an element of the form*

$$\sum_{i=0}^{k-1} a_i \gamma^{(i)}(t, x).$$

Proof. By Assumption 2 we infer that for small h, this discrete norm is also a norm over G_N. The uniform norm over $G_N[h]$ is dominated by the uniform norm over G_N. Since all finite dimensional norms are equivalent the the result follows from Theorem 1. ∎

Remark. Corollary 1 is still valid if the limit point is in R. For if we consider a bounded sequence $\{\sum_{i=1}^{m} a_{i\nu} \gamma(t_{i\nu}, x)\} \subset F[h]$ where $\lim_{\nu \to \infty} t_{i\nu} = \bar{t} \in R$ $(i = 1,..., m)$ then in the notation of Definition 1,

$$\sum_{i=1}^{m} a_{i\nu} \gamma(t_{i\nu}, x) = \sum_{i=1}^{m} b_{i\nu} \varphi(\lambda_{i\nu}, x).$$

Here

$$\varphi(\lambda_{i\nu}, x) \equiv h(t(\lambda_{i\nu}))\, \gamma(t(\lambda_{i\nu}), x),$$
$$b_{i\nu} = a_{i\nu} / h(t(\lambda_{i\nu})).$$

The result follows as in Corollary 1 by considering the sequence

$$\left\{ \sum_{i=1}^{m} b_{i\nu} \varphi(\lambda_{i\nu}, x) \right\},$$

where $\lim_{\nu \to \infty} \lambda_{i\nu} = \bar{\lambda}$ $(i = 1,..., m)$ and $\bar{\lambda} = \lambda(\bar{t})$.

If $\overline{F[h]}$ is the pointwise closure of $F[h]$ the following result holds.

THEOREM 2. *For small h*

$$\overline{F[h]} = \{f\colon [h] \to \text{Reals}\colon f = g + h + f_S\,;\ g \in G_j[h];\ h \in H_k[h];\ f_S \text{ is of the form generated in Assumption 4 with parameter } l;\ j + k + l \leqslant N\} \quad (3)$$

Proof. Let $\{f_\nu(x) = g_\nu(x) + h_\nu(x) + v_\nu(x)\}$ be a sequence in $F[h]$ which converges pointwise to $f(x)$. Here

$$g_\nu(x) = \sum_{i=1}^{j} a_{i\nu}\gamma(t_{i\nu}, x), \qquad \lim_{\nu\to\infty} t_{i\nu} = t_i \in T;$$

$$h_\nu(x) = \sum_{i=1}^{k} b_{i\nu}\gamma(t'_{i\nu}, x), \qquad \lim_{\nu\to\infty} t'_{i\nu} = t_i' \in R;$$

$$v_\nu(x) = \sum_{i=1}^{l} c_{i\nu}\gamma(t''_{i\nu}, x), \qquad \lim_{\nu\to\infty} t''_{i\nu} = t''_i \in S.$$

Consider any norm on $\{f\colon [h] \to \text{Reals}\}$. (We assume of course that h is small enough so that the assumptions on the family are valid.) We claim each sequence $\{g_\nu\}$, $\{h_\nu\}$ and $\{v_\nu\}$ is bounded in norm. If not by dividing each f_ν by $\max\{\| g_\nu \|, \| h_\nu \|, \| v_\nu \|\}$ we can assume that $\| f_\nu \| \to 0$ and

$$\max\{\| g_\nu \|, \| h_\nu \|, \| v_\nu \|\} = 1.$$

By going to a subsequence, which is not relabeled, it can be assumed that one of the three sequences say $\{g_\nu\}$ has the property $\| g_\nu \| = 1$ for all ν. By going to a subsequence again and using Theorem 1 and its corollary several times, it follows that $g_\nu \to g \in G_j[h]$ where $\| g \| = 1$. In addition by the remark after Theorem 1 it can be assumed that $h_\nu \to h \in H_k[h]$. Finally by the compactness of bounded functions over $[h]$, the condition that $v_\nu \to f_S$, where f_S is of the form generated in Assumption 4, can be secured. Clearly, over $[h]$

$$0 = g + h + f_S,$$

with $\| g \| = 1$. By Assumption 4 $f_S \equiv 0$. But then $g + h \equiv 0$ with $\| g \| = 1$ which contradicts Assumption 3. Therefore the three sequences $\{g_\nu\}$, $\{h_\nu\}$, $\{v_\nu\}$ are bounded. The techniques needed to show that the right hand side of (3) is a subset of $\overline{F[h]}$ is then obvious. The reverse containment is easily shown using the theory of differences [9]. ■

We now apply Theorem 2 to several examples and explicitly determine $\overline{F[h]}$ for these examples. Let

$$\gamma(t, x) = e^{tx} \qquad \text{and} \qquad T = (-\infty, \infty).$$

We shall show for sufficiently small h, the pointwise closure over $[h]$ of all functions of the form,

$$\sum_{i=1}^{N} a_i e^{t_i x},$$

consists of all functions which can be expressed as,

$$\sum_{i=1}^{r} \sum_{j=0}^{m_i} b_{ij} x^j e^{\lambda_i x} + f(x).$$

Here $f(x)$ is an arbitrary real-valued function on the grid $[h]$ which vanishes on the subset, $\{kh, (k+1)h, \ldots, 1-(m+1)h, 1-mh\}$, where

$$\sum_{i=1}^{r} (m_i + 1) + k + m \leqslant N.$$

Use will be made of the classical result that any exponential of the form

$$\sum_{i=1}^{k} a_i e^{\lambda_i x}, \qquad \text{where} \quad \sum_{i=1}^{k} |a_i| > 0, \tag{4}$$

has at most $k - 1$ zeros [8]. We assume for the remainder of the discussion on the exponentials that $[h]$ contains at least $3N$ points. Then, clearly Assumptions 1 and 2 are valid. The claim is made that both $\pm\infty$ are singular points. This can be seen by looking at the sequences $\{e^{nx}; n = 1, 2, \ldots\}$ and $\{e^{-nx}; n = 1, 2, \ldots\}$. In order to verify Assumptions 4 and to describe $\overline{F[h]}$ we require several lemmas.

LEMMA 1. *Let* $\{f_\nu(x) = \sum_{i=1}^{m} a_{i\nu} e^{t_{i\nu} x}\} \subset F[h]$ *converge pointwise to* $f(x)$ *over* $[h]$ *where* $m \leqslant N$ *and* $\lim_{\nu\to\infty} t_{i\nu} = \infty$ $(i = 1, \ldots, m)$. *Then* $f(z) = 0$ *for* $z \leqslant 1 - mh$ *and* $z \in [h]$.

Proof. Let E be the forward shift operator associated with h; that is,

$$(Eg)(x) = g(x + h).$$

If I is the identity operator then it follows easily from the commuting properties of these operators that for $z \in [h]$ and $z \leqslant 1 - mh$,

$$\prod_{i=1}^{m} (E - e^{ht_{i\nu}} I) f_\nu(z) = 0, \qquad \nu = 1, 2, \ldots. \tag{5}$$

Dividing both sides of (5) by $\prod_{i=1}^{m} e^{ht_{i\nu}}$,

$$\prod_{i=1}^{m} (E/e^{ht_{i\nu}}) - I) f_\nu(z) = 0. \tag{6}$$

Letting $\nu \to \infty$ in (6) we find

$$f(z) = 0.$$

This concludes the proof. ∎

LEMMA 2. *Let* $\{f_\nu(x) = \sum_{i=1}^{m} a_{i\nu} e^{t_{i\nu} x}\} \subset F[h]$ *converge pointwise to* $f(x)$ *over* $[h]$ *where* $\lim_{\nu\to\infty} t_{i\nu} = -\infty$ $(i = 1,..., m)$ *and* $m \leqslant N$. *Then* $f(z) = 0$ *for* $z \in [h]$ *and* $z \geqslant mh$.

Proof. Consider such a z. As before

$$\prod_{i=1}^{m} (E - e^{ht_{i\nu}} I) f_\nu(z_1) = 0,$$

where $z_1 = z - mh$. Letting $\nu \to \infty$ it follows that

$$E^m f(z_1) = 0;$$

that is $f(z) = 0$. ∎

We claim that any real-valued function on $[h]$ which vanishes on

$$[h] \sim \{0, h,..., (m-1)h\}$$

is the pointwise limit of a sequence of the form,

$$\left\{ f_\nu(x) = \sum_{i=1}^{m} a_{i\nu} e^{t_{i\nu} x};\ \lim_{\nu\to\infty} t_{i\nu} = -\infty;\ i = 1,..., m \right\}.$$

LEMMA 3. *Let* $\{b_1,..., b_m\}$ *be a set of m real numbers. There is a sequence*

$$\left\{ f_\nu(x) = \sum_{i=1}^{m} a_{i\nu} e^{t_{i\nu} x}: \lim_{\nu\to\infty} t_{i\nu} = -\infty;\ i = 1,..., m \right\}$$

such that $f_\nu((i-1)h) = b_i$ $(i = 1,..., m)$ *and* $f_\nu(z) \to 0$ *for*

$$z \in [h] \sim \{0,..., (m-1)h\}.$$

Proof. Select for each ν a set of m distinct numbers $\{t_{1\nu},..., t_{m\nu}\}$ such that $\lim_{\nu\to\infty} t_{i\nu} = -\infty$ $(i = 1,..., m)$. It is well known [8] that for each ν,

$$\{e^{t_{1\nu} x},..., e^{t_{m\nu} x}\}$$

forms a mth order Chebyshev system. Hence for each ν, there is a set of m real numbers $\{a_{1\nu},..., a_{m\nu}\}$ such that

$$f_\nu(x) = \sum_{i=1}^{m} a_{i\nu} e^{t_{i\nu} x}$$

has the property $f_\nu((i-1)h) = b_i$ $(i = 1,..., m)$. For $z \in [h] \sim \{0, h,..., (m-1)h\}$ using the same reasoning as in Lemma 2 it follows that $\lim_{\nu\to\infty} f_\nu(z) = 0$. ∎

Clearly then the following result is also valid.

LEMMA 4. *Let $\{b_1, ..., b_m\}$ be a set of m real numbers. Then there is a sequence*

$$\left\{f_\nu(x) = \sum_{i=1}^{m} a_{i\nu} e^{t_{i\nu} x} : \lim_{\nu\to\infty} t_{i\nu} = +\infty; (i = 1, ..., m)\right\}$$

such that $f_\nu(1 - (m - j)h) = b_j$ $(j = 1, ..., m)$ *and for* $z \in [h] \sim \{1 - (m - 1)h, 1 - (m - 2)h, ..., 1\}$,

$$\lim_{\nu\to\infty} f_\nu(z) = 0.$$

Remark. From Lemmas 1 and 2 it follows that if the sequence

$$\left\{g_\nu(x) = \sum_{i=1}^{m} a_{i\nu} e^{\lambda_{i\nu} x} + \sum_{i=1}^{k} b_{i\nu} e^{t_{i\nu} x}\right\}$$

is bounded over $[h]$ where $\lim_{\nu\to\infty} \lambda_{i\nu} = \infty$ $(i = 1, ..., m)$, $\lim_{\nu\to\infty} t_{i\nu} = -\infty$ $(i = 1, ..., k)$ with $k + m \leqslant N$, then each of the two sequences

$$\left\{\sum_{i=1}^{m} a_{i\nu} e^{\lambda_{i\nu} x}\right\} \quad \text{and} \quad \left\{\sum_{i=1}^{k} b_{i\nu} e^{t_{i\nu} x}\right\}$$

is bounded over $[h]$. (This is a direct consequence of the techniques used in Theorem 2 and the fact that $[h]$ contains at least $3N$ elements.) Further we can infer using the zero properties of an exponential [8] that Assumption 4 holds. Finally from Lemmas 1, 2, 3, 4 and Theorem 2 we conclude

$$\begin{aligned}\bar{F}[h] = \{f : f = g + f_{S0} + f_{S1}\,; g \in G_j[h];\\ f_{S0}(z) = 0 \text{ for } z \in [h] \sim \{0, h, ..., (k - 1)h\};\\ f_{S1}(z) = 0 \text{ for } z \in [h] \sim \{1 - (m - 1)h, ..., 1\}; j + k + m \leqslant N\}.\end{aligned}$$

The second family to be analyzed is generated by the Cauchy kernel,

$$\gamma(t, x) = 1/(t + x)$$

where $T = (0, \infty)$.

We shall prove that for sufficiently small h, the pointwise closure over $[h]$ of all functions of the form,

$$\sum_{i=1}^{N} \frac{a_i}{t_i + x},$$

consists of all functions,

$$\left(\sum_{i=0}^{r-1} a_i x^i \Big/ \prod_{i=1}^{k} (x + \lambda_i)\right) + \left(\sum_{j=0}^{m-2} b_j x^j \Big/ x^{m-1}\right).$$

Here the rational function on the right is replaced by an arbitrary number at $x = 0$. Further t_i and λ_i are both greater than zero, $r + m \leqslant N$ and $k \leqslant r$.

Assumptions 1 and 2 are satisfied for $(1/h) \geqslant N$; indeed in [6] it is shown that any function of the form

$$\sum_{i=1}^{k} \sum_{j=0}^{m_i} a_{ij} \gamma^{(j)}(t_i, x),$$

where $\sum\sum | a_{ij} | > 0$ and $t_i \in T$, has at most $\sum_{i=1}^{k} (m_i + 1) - 1$ zeros. $\bar{T} \sim T$ consists of the points $\{0, \infty\}$. We shall demonstrate later that "0" is a singular

LEMMA 5. *"∞" is a regular point for the Cauchy Kernel.*

Proof. Let $\lambda(t) = 1/t$ and $h(t) = t$. Then using the notation of Definition 1 with $\bar{t} = \infty$,

$$\varphi(\lambda, x) = 1/(\lambda x + 1),$$

and $\bar{\lambda} = \lim_{t\to\infty} (1/t) = 0$. A direct calculation reveals that

$$\varphi(0, x) = 1$$

$$\left.\frac{\partial^j \varphi(\lambda, x)}{\partial \lambda^j}\right|_{\bar{\lambda}=0} = (-1)^j j!\, x^j, \qquad j = 1, 2, \ldots, N-1.$$

By the remark after Corollary 1 and the theory of differences [9],

$$H_N = \{p(x) \colon \partial p \leqslant N - 1\}.$$

(Here we use the notation; ∂p means the degree of the polynomial p.) Our calculation reveals that the derivatives of φ are independent over $[h]$ for $(1/h) \geqslant N$. Since $G_N = \{q(x)/\prod_{i=1}^{m} (x + t_i), t_i \in T; \partial q \leqslant m - 1; m \leqslant N\}$, $G_N[h] \cap H_N[h] = \{0\}$ for $1/h \geqslant 2N$. Thus Assumption 3 is satisfied and "∞" is a regular point. ■

Although the results for the Cauchy Kernel are valid when $(1/h) \geqslant N$, in order to simplify the discussion we assume for the remainder of the analysis of this example that $(1/h) \geqslant 2N$. We now examine the point, "0".

LEMMA 6. *Let $\{f_\nu(x) = \sum_{i=1}^{m} a_{i\nu}/(t_{i\nu} + x); t_{i\nu} \in T; \lim_{\nu\to\infty} t_{i\nu} = 0; i = 1,\ldots, m \leqslant N\}$ be a sequence which converges to $f(x)$ over $[h]$. Then for $x \in [h] \sim \{0\}$,*

$$f(x) = p(x)/x^{m-1},$$

where $\partial p \leqslant m - 2$.

Proof. Consider $g_\nu(x) = \prod_{i=1}^{m} (t_{i\nu} + x) f_\nu(x)$ which is a polynomial of degree at most $m - 1$. Then for $x \in [h] \sim \{0\}$ it is well known [9] that

$$(E - I)^m g_\nu(x) = 0, \tag{7}$$

where E is again the forward shift operator associated with h and I is again the identity operator. Letting $\nu \to \infty$ in (7)

$$(E - I)^m g(x) = 0.$$

where $g(x) = x^m f(x)$. Again using a standard result on differences [9], $g(x)$ is a polynomial of at most degree $m - 1$. A straightforward argument based on the fact that $\{f_\nu(0)\}$ converges, yields the result that $g(0) = 0$. For $x \in [h] \sim \{0\}$ then $f(x) = p(x)/x^{m-1}$ where $\partial p \leqslant m - 2$. ∎

LEMMA 7. *For a given polynomial p of degree at most* $m - 2$ (*where* $m \leqslant N$) *and a real number* b_0, *there is a sequence*

$$\left\{f_\nu(x) = \sum_{i=1}^{m} \frac{a_{i\nu}}{x + t_{i\nu}};\ t_i \in T;\ \lim_{\nu\to\infty} t_{i\nu} = 0,\ i = 1,\dots, m\right\}$$

such that over $[h]$

$$\lim_{\nu\to\infty} f_\nu(x) = f(x),$$

where $f(0) = b_0$ *and* $f(x) = p(x)/x^{m-1}$ *for* $x \in [h] \sim \{0\}$.

Proof. For each ν, select $0 < t_{1\nu} < t_{2\nu} < \cdots < t_{m\nu}$ such that $\lim_{\nu\to\infty} t_{i\nu} = 0$ ($i = 1,\dots, m$). Select $m - 1$ distinct points $\{x_1,\dots, x_{m-1}\} \subset [h] \sim \{0\}$. $p(x)$ is uniquely determined by the values b_i it takes on at x_i ($i = 1,\dots, m - 1$). Now since $\{1/(x + t_{i\nu}),\dots, (1/x + t_{m\nu})\}$ for each ν is a m-th order Chebyshev system [6], choose $\{a_{1\nu},\dots, a_{m\nu}\}$ so that

$$f_\nu(x) \equiv \sum_{i=1}^{m} \frac{a_{i\nu}}{x + t_{i\nu}}$$

has the properties,

$$f_\nu(0) = b_0,$$

$$f_\nu(x_i) = x_i b_i \Big/ \prod_{i=1}^{m} (x_i + t_{i\nu}), \qquad i = 1,\dots, m - 1.$$

Hence,

$$f_\nu(x) = p_{m-1}^{(\nu)}(x) \Big/ \prod_{i=1}^{m} (x + t_{i\nu}),$$

where

$$\partial p_{m-1}^{(\nu)} \leqslant m-1$$

$$p_{m-1}^{(\nu)}(0) = b_0 \prod_{i=1}^{m} t_{i\nu},$$

$$p_{m-1}^{(\nu)}(x_j) = x_j b_j, \qquad j = 1,..., m-1.$$

Each $p_{m-1}^{(\nu)}$ is bounded at these m points and $\lim_{\nu\to\infty} p_{m-1}^{(\nu)}(0) = 0$. Hence by going to a subsequence we can assume

$$p_{m-1}^{(\nu)} \to p_{m-1} \qquad \text{uniformly}$$

where

$$p_{m-1}(0) = 0,$$

$$p_{m-1}(x_i) = x_i p(x_i), \qquad i = 1,..., m-1.$$

Clearly, $f_\nu(x) \to xp(x)/x^m = p(x)/x^{m-1}$ for $x \in [h] \sim \{0\}$, and $f_\nu(0) \to b_0$. ∎

Remark. It is easy to check using Lemmas 5 and 6 that in order for Assumption 4 to hold it is sufficient to demonstrate that if the equality

$$p(x) \Big/ \prod_{i=1}^{k} (x + t_i) = q(x)/x^{m-1}$$

is valid over $[h] \sim \{0\}$ where $t_i > 0$, $\partial p \leqslant (k-1)$, $k \leqslant N$, and $\partial q \leqslant m - 2 \leqslant N - 2$, then $p \equiv q \equiv 0$. But if $q \not\equiv 0$, x^{m-1} must be a factor of q, a polynomial of at most degree $m - 2$, a contradiction. Thus Assumption 4 holds and "0" is a singular point.

From Theorem 2 and Lemmas 5, 6, and 7 it follows readily that

$$\overline{F[h]} = \Big\{ \Big(\sum_{i=0}^{r-1} a_i x^i \Big/ \prod_{i=1}^{k} (x + t_i) \Big) + r(x);\ t_i > 0,\ i = 1,..., k;$$

$$r(x) = q(x)/x^{m-1} \text{ over } [h] \sim \{0\},\ \partial q \leqslant m-2;\ k \leqslant r;\ r + m \leqslant N \Big\}.$$

Continuous Case

For any L_p norm $1 \leqslant p \leqslant \infty$ we consider the existence question. For such a norm $\| \ \|$, where

$$\|f\| = \left(\int_0^1 |f|^p \right)^{1/p} \qquad \text{for} \quad 1 \leqslant p < \infty,$$

$$\|f\| = \max_{x \in [0,1]} |f(x)| \qquad \text{for} \quad p = \infty,$$

we have their discrete analog over $[h]$; i.e.,

$$\|f\|_{[h]} = h\Big(\sum_{x\in[h]} |f(x)|^p\Big)^{1/p} \qquad \text{for} \quad 1 \leqslant p < \infty;$$

$$\|f\|_{[h]} = \max_{x\in[h]} |f(x)| \qquad \text{for} \quad p = \infty.$$

Note that in order to avoid some messy notation in our previous dicussion we have used both the point 0 and 1 in defining the L_p norms. For any subset J of [0, 1] we define

$$\| f\|_{[h]\cap J} = \| g \|_{[h]}$$

with $g(x) = f(x) \cdot \chi(x)$ where χ is the characteristic function associated with the set J. We need one further assumption to allow us to proceed from the discrete problem to the continuous problem.

As will be seen in the proof of Theorem 4, in going to the continuous case, we need not consider singular functions directly, but need only consider sequences in F of the sort $f_\nu(x) + v_\nu(x)$ where every parameter sequence $\{\lambda_{j\nu}\}$ involved in $f_\nu(x)$ converges to some point $\lambda_j \in T \cup R$ while every parameter sequence $\{t_{i\nu}\}$ involved in $v_\nu(x)$ converges to some point $t_i \in S$.

Assumptions 5 below states that as we pass to the limit the contributions of any singular functions (represented by the v_ν's) will disappear.

ASSUMPTION 5. *For any sequence of equally spaced grids*

$$\{[h_\nu]: \lim_{\nu\to\infty} h_\nu = 0\}$$

and any sequence $\{v_\nu(x) = \sum_{i=1}^{k} a_{i\nu}\gamma(t_{i\nu}, x)\} \subset F$ *where*

(a) $\lim_{\nu\to\infty} t_{i\nu} = t_i \in S$, $i = 1,..., k \leqslant N$,

(b) $\| v_\nu \|_{[h_\nu]} \leqslant K$ *for all* ν,

the following are valid.

There are subsequences (which we do not relabel) and a sequence of closed sets $\{F_j\}$ *with the properties*:

(1) *Each* F_j *is the union of a finite number of closed intervals.*

(2) $F_j \subset F_{j+1} \subset [0, 1]$ *for all* j.

(3) *The complement in* [0, 1] *of* $\bigcup_{j=1}^{\infty} F_j$ *has measure zero.*

(4) *For each* j, $\lim_{\nu\to\infty} \| v_\nu \|_{F_j\cap[h_\nu]} = 0$.

THEOREM 3. *Consider a sequence of functions* $\{g_\nu\}$ *of the form*

$$g_\nu = f_\nu + v_\nu \tag{8}$$

with $g_\nu \in \bar{F}[h_\nu]$ *and* $h_\nu \to 0$ *with the properties*:

(a) $f_\nu(x) = \sum_{i=1}^{k} a_{i\nu}\gamma(t_{i\nu}, x)$ *where* $t_{i\nu} \to t_i \in T \cup R$ $i = 1,..., k \leqslant N$.

(b) *The sequence* $\{v_\nu\}$ *is of the form described in Assumption* 5.

(c) *For all* ν, $\| g_\nu \|_{[h_\nu]} \leqslant K$.

Then the two sequences $\{\| f_\nu \|_{[h_\nu]}\}$ *and* $\{\| v_\nu \|_{[h_\nu]}\}$ *are each bounded.*

Proof. Assume to the contrary. It is easy to check using the triangle inequality that this implies that the sequence $\{\| f_\nu \|_{[h_\nu]}\}$ is unbounded. Hence by dividing both sides of (8) by $\| f_\nu \|_{[h_\nu]}$, it can be assumed that $\lim_\nu \| g_\nu \|_{[h_\nu]} = 0$; $\| f_\nu \|_{[h_\nu]} = 1$, and $\lim_\nu \| v_\nu \|_{[h_\nu]} = 1$. Using Corollary 1 and the remark following, we infer that $f_\nu \to f$ in the uniform norm over [0, 1]. (Here again we do not relabel subsequences.) Using the standard properties of the Riemann Integral for $1 \leqslant p < \infty$, we find for all $1 \leqslant p \leqslant \infty$ that $\lim_\nu \| f - f_\nu \| \to 0$ with $\| f \| = 1$. Now applying Assumption 5 it follows that a subsequence of the $\{v_\nu\}$ and a sequence of closed sets $\{F_j\}$ satisfying properties 1, 2, and 3 of Assumption 5 exist so that for each j

$$\lim_{\nu \to \infty} \| v_\nu \|_{F_j \cap [h_\nu]} = 0.$$

(Again we do not relabel subsequences.) Select an F_k such that

$$\| f \|_{F_k} = \alpha > 0,$$

where $\| f \|_{F_k} = \| \chi f \|$ with χ being the characteristic function of F_k. Since $\| f_\nu + v_\nu \|_{[h_\nu]} \to 0$ and

$$\| f_\nu + v_\nu \|_{[h_\nu] \cap F_k} \geqslant \| f_\nu \|_{[h_\nu] \cap F_k} - \| v_\nu \|_{F_k \cap [h_\nu]}$$

by going to the limit we reach the contradiction that

$$0 \geqslant \alpha - 0 > 0.$$

Hence the two sequences are bounded. ∎

Consider any $\rho(x) \in C[0, 1]$. We shall show that in the continuous norm over [0, 1] $\rho(x)$ has a closest point from the set

$$\bar{F} = \{f = g + h; g \in G_k\,; h \in H_k\,; j + k \leqslant N\}.$$

Further our proof will demonstrate $\bar{F}$ is the uniform closure of F as follows. From the theory of differences [9], we can deduce that $\bar{F}$ is a subset of the uniform closure of F. Our existence proof will yield the reverse inclusion. For $\rho \in L_p[0, 1]$ $(1 \leqslant p \leqslant \infty)$ our methods can be used also to prove existences. This is accomplished by noting the step functions are dense in $L_p[0, 1]$ and defining ρ correctly on a set of measure zero.

THEOREM 4. *For each $\rho(x) \in C[0, 1]$, there is a $f \in \bar{F}$ such that*

$$\| \rho - f \| = \text{distance}(\rho, \bar{F}) \equiv \inf\{\| \rho - g \| : g \in \bar{F}\}.$$

Here $\| \ \|$ is the L_p norm for $1 \leqslant p \leqslant \infty$.

Proof. Consider a sequence of grids $\{[h_\nu]\}$ such that $h_\nu \to 0$. Let g_ν be the best approximation to ρ from $\bar{F}[h_\nu]$; i.e.,

$$\| \rho - g_\nu \|_{[h_\nu]} = \inf\{\| \rho - g \|_{[h_\nu]} : g \in \bar{F}[h_\nu]\}.$$

Since $\bar{F}[h]$ is the closure of $F[h]$ there is for each ν a function $\hat{g}_\nu \in F[h_\nu]$ of the form

$$\hat{g}_\nu(x) = f_\nu(x) + v_\nu(x)$$

with f_ν and v_ν having properties (a) and (b) respectively of the hypotheses of Theorem 3. Further,

$$\| \rho - \hat{g}_\nu \|_{[h_\nu]} \leqslant \| \rho - g_\nu \|_{[h_\nu]} + \frac{1}{\nu}.$$

For any $g \in \bar{F}$,

$$\| \rho - \hat{g}_\nu \|_{[h_\nu]} + \frac{1}{\nu} \leqslant \| \rho - g \|_{[h_\nu]},$$

$$\lim_{\nu \to \infty} \| \rho - g \|_{[h_\nu]} = \| \rho - g \|.$$

Since $\| \rho \|_{[h_\nu]} \to \| \rho \|$ in addition it then follows that the sequence $\{\| \hat{g}_\nu \|_{[h_\nu]}\}$ is bounded. By Theorem 3 then each of the sequences $\{\| f_\nu \|_{[h_\nu]}\}$ and $\{\| v_\nu \|_{[h_\nu]}\}$ is bounded. Now applying the techniques of Theorem 3 we infer (again not relabelling subsequences) that for a sequence of closed sets $\{J_k\}$ with the properties assigned in Assumption 5 the following is valid:

For each k, $\lim_{\nu \to \infty} \| v_\nu \|_{[h_\nu] \cap J_k} = 0$; $f_\nu \to f$ in the uniform norm where $f \in \bar{F}$.

The claim is made that

$$\| \rho - f \| = \text{distance}(\rho, \bar{F}).$$

Consider any $g \in \bar{F}$. For each k,

$$\begin{aligned} \| \rho - g \|_{[h_\nu]} &\geqslant \| \rho - \hat{g}_\nu \|_{[h_\nu]} - \frac{1}{\nu} \\ &\geqslant \| \rho - \hat{g}_\nu \|_{[h_\nu] \cap J_k} - \frac{1}{\nu} \\ &\geqslant \| \rho - f_\nu \|_{[h_\nu] \cap J_k} - \| v_\nu \|_{[h_\nu] \cap J_k} - \frac{1}{\nu}. \end{aligned}$$

The left most quantity above converges to $\| \rho - g \|$. The right most quantity above converges to $\| \rho - f \|_{J_k}$. Thus

$$\| \rho - g \| \geqslant \| \rho - f \|_{J_k}.$$

Letting $k \to \infty$,

$$\| \rho - g \| \geqslant \| \rho - f \|.$$

Since g was an arbitrary element of $\bar{F}$ the proof is complete. ∎

The following result is then clear.

COROLLARY 2. *Let* $\{[h_\nu]\}$ *be a sequence of grids with* $h_\nu \to 0$. *Let* $f_\nu(x)$ *be the best approximation from* $\overline{F[h_\nu]}$ *to* $g \in C[0, 1]$ *under the norm* $\| \ \|_{[h_\nu]}$ *with* $f_\nu(x)$ *having the form*

$$f_\nu(x) = \sum_{i=1}^{m} \sum_{j=0}^{m_i} a_{ij\nu} \gamma^{(j)}(t_{i\nu}, x) + \sum_{i=1}^{k} \sum_{j=0}^{k_i} b_{ij\nu} \varphi^{(j)}(\lambda_i, x)$$

$$+ f_{S_\nu}(x) + \sum_{j=1}^{r} c_{i\nu} \gamma^{(j)}(t'_{j\nu}, x)$$

where

$$\lim_\nu t_{i\nu} = t_i \in R \cup T, \qquad i = 1,..., m;$$

$$\lambda_i \in R, \qquad i = 1,..., k;$$

$$\lim_\nu t'_{j\nu} = t_j' \in S \qquad j = 1,..., r.$$

$f_{S_\nu}(x)$ *is a singular function for the grid* $[h_\nu]$.

Then for some subsequence of $\{f_\nu\}$ *which is not relabeled,*

$$\sum_{i=1}^{m} \sum_{j=0}^{m_i} a_{ij\nu} \gamma^{(j)}(t_{i\nu}, x) + \sum_{i=1}^{k} \sum_{j=0}^{m_i} b_{ij\nu} \varphi^{(j)}(\lambda_i, x) \to f(x) \in \bar{F}.$$

Here $f(x) = \bar{f}(x) + \hat{f}(x)$ *with* $\bar{f}(x) \in G_j$, $\hat{f}(x) \in H_k$. *Further* $j + k \leqslant N$ *and* $j < N$ *if* $\sum_{t_i \in T} (m_i + 1) < N$, *where in both cases the convergence is in the uniform norm over* $[0, 1]$. *Finally* f *is a best approximation to* g *from* $\bar{F}$.

In order to establish that Assumption 5 is satisfied for our examples the following definition will be useful.

DEFINITION 2. A sequence of functions $\{g_\nu\} \subset C^2[0, 1]$ is said to have *property S* where S is some positive integer if $d^2 g_\nu(x)/dx^2 \not\equiv 0$ implies $d^2 g_\nu(x)/dx^2$ has at most S zeros.

Families with this property were introduced in [4]. It can be easily shown using the techniques developed in that paper that the following theorem is valid.

THEOREM 5. *Consider any sequence* $\{v_\nu\} \subset C^2[0, 1]$ *with property S which in addition are bounded in the* $\| \ \|_{[h_\nu]}$ *where* $h_\nu \to 0$; i.e., *there exists a* $K > 0$ *such that for all* ν

$$\| v_\nu \|_{[h_\nu]} \leqslant K.$$

Then there exists a sequence of closed sets $\{F_j\}$ *and a subsequence of the* $\{v_\nu\}$ *so that statements* 1, 2, *and* 3 *of Assumption* 5 *are valid for* $\{F_j\}$. *Further there is a function v so that for each j*,

$$\lim_{\nu\to\infty} \max_{x\in F_j} | v_\nu(x) - v(x)| \to 0.$$

(*Again we have not relabeled the subsequence.*) *In addition each* F_j *is the union of at most* $3S + 4$ *disjoint intervals.*

Note that the exponential family has property S with $S = N - 1$. Also the rational fraction family generated by the Cauchy kernel has property S with $S = 3N - 1$.

THEOREM 6. *The exponential family satisfies Assumption* 5.

Proof. It suffices to show if

$$v_\nu(x) = \sum_{i=1}^{k} e^{t_{i\nu}x} + \sum_{j=1}^{m} b_{j\nu} e^{t'_{j\nu}x},$$

where $t_{i\nu} \to \infty, i = 1,..., k; t'_{j\nu} \to -\infty, j = 1,..., m$ and if F is a subinterval of $[0, 1]$ with the property that for some $v(x)$

$$\lim_{\nu\to\infty} \max_{x\in F} | v_\nu(x) - v(x)| = 0.$$

then $v(x) = 0$ for $x \in F \sim \{0, 1\}$. For such an x, choose $h > 0$ small enough so that $x_i \equiv (x - ih) \in F$, $i = 1,..., m$ and $y_j \equiv (x + jh) \in F$ $(j = 1,..., k)$. Then as in Lemmas 1 and 2

$$\prod_{i=1}^{k} \left(\frac{E}{e^{t_{i\nu}h}} - I\right) \prod_{j=1}^{m} (E - e^{t'_{j\nu}h} I)\, v_\nu(x_m) = 0.$$

Letting $\nu \to \infty$ we find

$$E^m v(x_m) = 0.$$

But $E^m v(x_m) = v(x)$. ∎

THEOREM 7. *The rational fraction family generated by the Cauchy kernel satisfies Assumption* 5.

Proof. Let $\{v_\nu(x) = \sum_{i=1}^{N} (a_{i\nu}/(x + t_{i\nu})), \lim_{\nu\to\infty} t_{i\nu} = 0, i = 1,..., N; t_{i\nu} > 0\}$ be a sequence such that $\| v_\nu(x)\|_{[h_k]} \leqslant c$ for all k where $h_k \to 0$. By using property S there exists a sequence of closed sets $\{F_k\}$ and a subsequence of the $\{v_\nu\}$ which satisfy the conclusion of Theorem 5 with $v(x)$ being the limit function. Note over each F_k, the appropriate L_p norm of v is bounded by c. Using the ideas of Lemma 6 it follows that

$$v(x) = p(x)/x^{N-1},$$

where $p(x)$ is a polynomial of at most degree $N - 2$ over each $F_k \sim \{0\}$. Our argument doesn't preclude the possibility that in each of the subintervals that form F_k, $p(x)$ is a different polynomial. We can rule out this possibility by noting that the number of subintervals which form F_k is bounded by a number independent of k. Thus for large k; our difference equation technique allows us to go from one subinterval to the other, thus yielding the same polynomial. Since v is bounded over each of the F_k in norm by c and degree of p is less than $N - 2$ it follows that

$$v(x) = 0$$

if $x \neq 0$. ∎

UNIFORM CONVERGENCE AND NORMALITY

In this section we show how our difference equation techniques can be used to obtain uniform convergence on closed subintervals. Our first result generalizes a theorem of Schmidt [11], who employed the continuous uniform norm on the exponential family.

THEOREM 7. *Let* $\{f_\nu(x) = \sum_{i=1}^{k} a_{i\nu} e^{t_{i\nu}x} + \sum_{j=1}^{l} b_{j\nu} e^{t_{j\nu}x}\}$ *be a sequence where* $t_{i\nu} \to -\infty$ $(i = 1,..., k)$ *and* $t_{j\nu} \to \infty$ $(j = 1,..., l)$. *Suppose there exists a constant* $K > 0$ *and a sequence of grids* $\{[h_\nu]\}$ *on* $[0, 1]$ *with the property that* $h_\nu \to 0$ *and* $\|f_\nu\|_{[h_\nu]} \leqslant K$. *Then a subsequence of* $\{f_\nu\}$ *converges uniformly to zero on every closed subinterval of* $(0, 1)$. *Here* $\| \ \|_{[h]}$ *is any discrete* L_p *norm on* $[h]$ *for some fixed* p *where* $1 \leqslant p \leqslant \infty$.

Proof. Let J be an arbitrary closed subinterval of (0, 1). Now $\{f_\nu\}$ has property S with $S = k + l - 1$, so by Theorem 5 a subsequence can be extracted (which is not relabeled) and a sequence of closed nested sets $\{A_m\}$ where $A_m = \bigcup_{i=1}^{L} [a_{im}, b_{im}] \equiv \bigcup_{i=1}^{L} I_{im}$ with $I_{jm} \cap I_{km} = \varphi$ if $j \neq k$,

$L \leqslant 3(k+l-1)+4$ such that $\mu(A_m) \to 1$ and f_ν converges uniformly on each A_m. Furthermore we may assume that

$$a_{1m} < b_{1m} < a_{2m} < \cdots < a_{Lm} < b_{Lm}$$

for all m. Choose m so large that

$$d \equiv \max_{1 \leqslant i \leqslant L-1} \{a_{i+1,m} - b_{im}\} < \frac{1}{l+K+1} \min_{1 \leqslant i \leqslant L} \{b_{im} - a_{im}\} \tag{8}$$

and $J \subset [a_{1m}, b_{Lm}]$. An arbitrary point x of the open interval (b_{im}, a_{i+1}, m) can be expressed as the $k+1$ member of a uniform grid of width d containing $(l+k+1)$ elements where by (8) every element of the grid excluding x is in A_m. Note that since $\{f_\nu\}$ converges uniformly on A_m if we set

$$\|f\|_{A_m} \equiv \max_{x \in A_m} |f(x)|$$

the sequence $\{\|f_\nu\|_{A_m}\}$ is bounded. Let E be the shift operator associated with d and let x_0 denote the first member of the grid which includes x. Then for each ν

$$\prod_{i=1}^{k} (E - e^{t_{i\nu} d} I) \prod_{j=1}^{l} (e^{-t'_{j\nu} d} E - I) f_\nu(x_0) = 0.$$

This relation can be expressed as

$$\sum_{i=1}^{l+k} \alpha_{i\nu} E^i f_\nu(x_0) = 0,$$

where $\lim_{\nu\to\infty} \alpha_{i\nu} = 0$ $(i \neq k)$ and $\lim_{\nu\to\infty} |\alpha_{k\nu}| = 1$. Thus

$$|\alpha_{k\nu}|\, |f_\nu(x_0 + kd)| \leqslant \sum_{\substack{i=1 \\ i \neq k}}^{k+l} |\alpha_{i\nu}|\, |f_\nu(x_0 + id)|.$$

As remarked previously, $x_0 + id \in A_m$ if $i \neq k$. Hence,

$$|f(x)| \leqslant |\beta_\nu|\, \|f_\nu\|_{A_m} \tag{9}$$

where $|\beta_\nu| \to 0$ and β_ν doesn't depend on x or m. Using the same difference equation techniques on the points of A_m we can show $f_\nu(x)$ converges pointwise to zero in the interior of A_m. Thus the uniform convergence on A_m and (9) together imply that $\{f_\nu\}$ converges uniformly to zero on J. ∎

COROLLARY 3. *Let* $\{f_\nu(x) = \sum_{i=1}^{N} a_{i\nu} e^{t_{i\nu} x}\}$ *be a sequence in* F *such that there exists a constant* $K > 0$ *and a sequence of grids* $\{[h_\nu]\}$ *with the property* *that* $h_\nu \to 0$ *and* $\|f_\nu\|_{[h_\nu]} \leqslant K$. *Then there is a subsequence of* $\{f_\nu\}$ *converging to some* $f \in \bar{F}$ *uniformly on every closed subinterval of* (0, 1).

Proof. The corollary easily follows from Theorem 7, Theorem 3, and Corollary 1 (including the remark following Corollary 1). ∎

In [4] it was shown that Theorem 5 was valid for the L_p norms $(1 \leqslant p \leqslant \infty)$. Hence Theorem 7 is valid for these norms. For our rational functional family using similar difference equation methods and the standard compactness properties of rational functions [10] the following theorem can be readily established.

THEOREM 8. *Let* $\{f_\nu(x) = \sum_{i=1}^{k} (a_{i\nu}/(t_{i\nu} + x))\} \subset F$ *where* $\lim_{\nu\to\infty} t_{i\nu} = 0$ $(i = 1,..., k)$. *Further there is a sequence of grids* $\{[h_\nu]\}$ *where* $h_\nu \to 0$ *and a* $K \geqslant 0$ *such that* $\|f_\nu\|_{[h_\nu]} \leqslant K$. (*Here of course we are employing a discrete* L_p *norm*). *Then there is a subsequence of* $\{f_\nu\}$ *which converges uniformly to zero on any closed subinterval of* (0, 1].

This result is also valid for the L_p norms $(1 \leqslant p \leqslant \infty)$.

COROLLARY 4. *Let* $\{f_\nu(x) = \sum_{i=1}^{k} (a_{i\nu}/(t_{i\nu} + x))\} \in F$, *where* $\lim_\nu t_{i\nu} = 0$ $i = 1,..., k' \leqslant k$. *Further assume there is a sequence of grids* $\{[h_\nu]\}$ *where* $h_\nu \to 0$ *and a* $K > 0$ *such that* $\|f\|_{[h_\nu]} \leqslant K$. *Then there is a subsequence of* $\{f_\nu\}$ *which converges uniformly on any closed subinterval of* (0, 1] *to a function of the form,*

$$\sum_{i=0}^{k-k'-1} a_i x^i \Big/ \prod_{j=1}^{m} (x + t_i),$$

where $t_i > 0$, $m \leqslant k - k'$ (*in case* $k = k'$, *the function is identically zero*).

Consider any L_p norm, $\| \ \|$, $(1 \leqslant p \leqslant \infty)$ on $C[0, 1]$.

DEFINITION. We say a $g \in C[0, 1]$ is normal if any best approximation to g from $\bar{F}$ has the form,

$$f(x) = \sum_{i=1}^{m} \sum_{j=0}^{m_i} a_{ij} \gamma^{(j)}(t_i \ , x), \tag{9}$$

where $\sum_{i=1}^{m} (m_i + 1) = N$ and $a_i \neq 0$ $(i = 1,..., m)$.

A consequence of Corollary 2 is the following.

THEOREM 9. *Let* $g \in C[0, 1]$ *be normal. Then for sufficiently small* h *any best approximation to* g *over* $[h]$ *from* $\overline{F[h]}$, *using as the norm the discrete analog of the* L_p *norm, is of the form* (9). *In addition, if the best approximation to* g *from* $\bar{F}$ *is unique, let* $T_h g$ *be any best approximation to* g *from* $\overline{F[h]}$. *Then*

$h_\nu \to 0$ implies $T_{h_\nu} g$ converges to f uniformly where f is the continuous best approximation to g from $\bar{F}$.

Proof. If there is a sequence $\{h_\nu\} \to 0$ such that a best approximation to g from $F[h_\nu]$ is not of the form (9), then using Corollary 2 we see that a best approximation to g exists which is not normal, a contradiction. Further assume the best approximation f, to g is unique. If $T_{h_\nu}(g) \not\to f$ there is a subsequence which we do not relabel, such that

$$\| T_{h_\nu} g - f \| \geqslant \epsilon > 0$$

for all ν. But again by Corollary 2 and the uniqueness of f, a subsequence converges to f, a contradiction. ∎

A theorem of Hobby and Rice [1] yields the result for both the exponential and rational function families that $1 < p < \infty$ and $g \in L_p[0, 1] \sim \bar{F}$ imply g is normal.

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